DEUTERIUM ARRAY MEMO #060

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November 1, 2004

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To: Deuterium Array Group

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Subject: Combining data from separate regions to improve the detection probability

Since we are looking at low SNRs in the spectra it has been suggested that we might combine the data from adjacent beams to improve the SNR. If beams pointed at different regions of the sky are combined we need to investigate the noise correlation statistics.

1] Receiver noise correlation beam width

Simultaneous beams are formed from the vector addition of the same receiver noise so that

$$b_0 = \sum_j e^{i\phi_j} n_j$$
$$b_1 = \sum_j e^{i\theta_j} n_j$$

and hence the correlation of the receiver noise between beams is

$$\left\langle b_0 b_1^* \right\rangle = \sum_j \sum_k e^{i(\phi_j - \theta_k)} \left\langle n_j n_k^* \right\rangle$$
$$= \sum_j e^{i(\phi_j - \theta_i)} \text{ since } n_j n_k^* = 1 \;\; j = k$$
$$= 0 \;\; j \neq k$$

where the phase ϕ_j *and* θ_j are used to steer the beams. The receiver noise correlation beam shape is therefore just the square root of the beam power response which is

$$|b_0|^2 = \left| \sum_j e^{i\phi_j} \right|^2$$

since the receiver noise from the individual receiver channels is uncorrelated. If beam is Gaussian then the receiver correlation beamwidth will be $\sqrt{2}$ times the power beamwidth.

2] Sky noise correlation beamwidth

If 2 beams are looking at a uniform sky brightness the sky noise correlation between them is given by the auto convolution of the beam pattern since:

$$\left\langle b_0 b_1^* \right\rangle = \int_{-\infty}^{+\infty} b\left(x\right) b\left(x-t\right) dx$$

where *b(x)* is the beam response in one dimension. For a Gaussian beam pattern

$$b\left(x\right) = e \frac{-4\left(\log_e 2\right) x^2}{w^2}$$

where w = full width at half power points.

And the correlation beam pattern is

$$b\left(t\right) = e \frac{-4\left(\log_e 2\right) t^2}{2w^2}$$

where t = separation between the beams

If the spectra from N beams are added with equal weighting the one sigma noise is

$$\sigma = \left[\frac{\sum_i \sum_j n_i n_j^*}{N^2} \right]^{1/2}$$

so that if the beams are uncorrelated the noise decreases by $1/\sqrt{N}$ otherwise if the correlation beam pattern is Gaussian and the beams are uniformly spaced in a line

$$\sigma = \left[\frac{\sum_i \sum_j e \frac{-4\left(\log e2\right)\left(i-j\right)^2 t^2}{2w^2}}{N^2} \right]^{1/2}$$

where t=spacing between the beams when N is large the expression simplifies to

$$\sigma = \left[\frac{\sum_i e \frac{-4\left(\log e^2\right) i^2 t^2}{2w^2}}{N} \right]^{1/2}$$

and sigma approaches a limit of $1/\sqrt{N}$ when the beams are spaced about 1.5 beamwidths apart and adding more beams to cover the same total span will not significantly improve the SNR. This happens because the beams become correlated the more closely they are spaced.

3] Comparing 5 vs 3 beams

A specific case considered is 3 beams with 12 degrees between beams and 5 beams with 6 degrees between beam centers. The total span is 24 degrees in each case. If we first assume that the expected D1 spectrum is the same for each beam

For 3 beams:

$$\sigma_3 = \left(3 + 4e^{-a^2} + 2e^{-4a^2}\right)^{1/2} \Big/ 3 = 0.687$$

$$\sigma_5 = \left(5 + 8e\frac{-a^2}{4} + 6e^{-a^2} + 4e\frac{-9a^2}{4} + 2e^{-4a^2}\right)^{1/2} \Big/ 5 = 0.723$$

where $a = 2^{1/2}\left(\log_e 2\right)^{1/2}\left(t/w\right)$

t = 12 degrees

w = beamwidth = 13 degrees

The noise is higher for the case of 5 beams because uniform weighting is not optimal in the presence of correlation between measurements. Now consider the case of 3 beams spaced 12 degrees apart with unequal signals in each beam. For simplicity consider the symmetrical case of a difference between the center and the outer beams. In this case the

$$\left(\frac{S}{N}\right)^2 = \left(1 + 2wb\right)^2 \Big/ \left(1 + 2w^2 + 4we^{-a^2} + 2w^2e^{-4a^2}\right)$$

where b = relative strength of signal in the outer beams

w = relative weight applied to the outer beams

The solution for optimum SNR is given by

$$w = \left(b - e^{-a^2}\right) \Big/ \left(1 - 2be^{-a^2} + e^{-4a^2}\right) = 1.75 \quad \text{when } b = 1$$

and in this case $\sigma = 0.678$ which is slightly better than the case of uniform weighting. If there is a 25% weaker signal expected from the outer beams then b=0.75 and the optimum weight of 0.8 results in a value of sigma of 0.822 for the optimum combination of the 3 beams. The actual case of optimizing the detection of the D1 line from the combined observations of several simultaneous beams is even more complex as the expected spectrum is different for each beam.

In this case we need to consider the weighted least squares solution to a particular line profile. The sigma spared in the least squares is given (from memo #56) by

$$\left\langle (\hat{s}-s)^H (\hat{s}-s) \right\rangle = \left(A^H wA\right)^{-1} A^H w \left\langle nn^H \right\rangle w^H A \left(A^H wA\right)^{-1}$$

If the noise is uncorrelated from frequency to frequency and from beam to beam this simplifies to $\sigma^2 = \left(A^H wA\right)^{-1} \sigma_0^2$ because the autocorrelation of the noise is a delta function. The expression can also be simplified if only one parameter is estimated and the weighting is uniform to

$$\sigma^2 = \frac{\sum_i \sum_j f_i f_j \left\langle n_i n_j \right\rangle}{\left(\sum f^2\right)^2} \sigma_0^2$$

where *f* is the expected line profile.

If the beams are correlated with coefficient C the one sigma error is approximately degraded by a factor of $(1+2C)^{1/2}$ compared with uncorrelated beams. For beams 12 degrees apart the correlation C is about 0.28 so that any calculation of the one sigma noise from the covariance matrix becomes $\sigma = \left[\left(A^H wA\right)^{-1}(1+2C)\right]^{1/2} \sigma_0$ needs to be degraded by a factor of $(1.56)^{1/2}$=1.25.

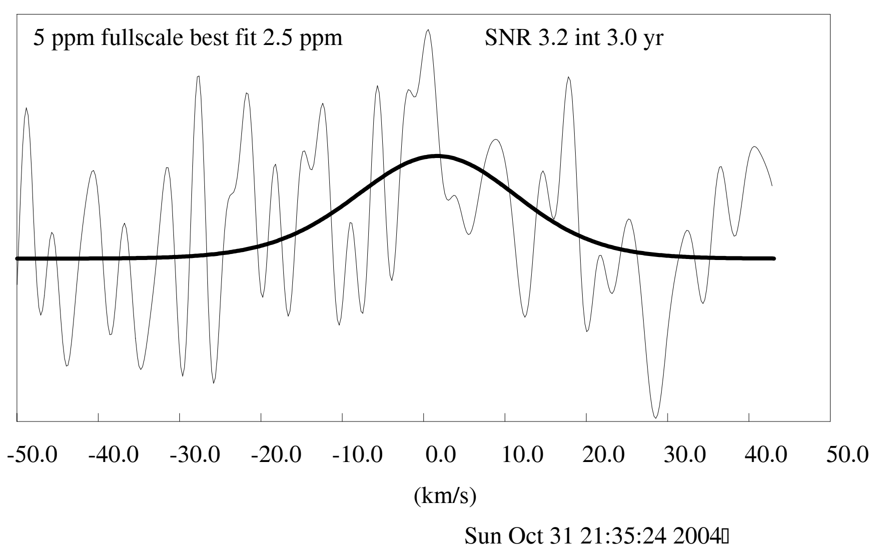

Figure 1 shows the fit to the beams at Galactic longitudes 171 and 183 degrees combined for data through day 299 of 2004. The individual beams give amplitudes of 2.6 and 2.4 ppm and SNRs of 3.3 and 2.4 respectively. Without correction for the correlation between beams the SNR of the fit to expected profiles for beams combined is 4.0. After application of the correlation correction the SNR drops to 3.2.