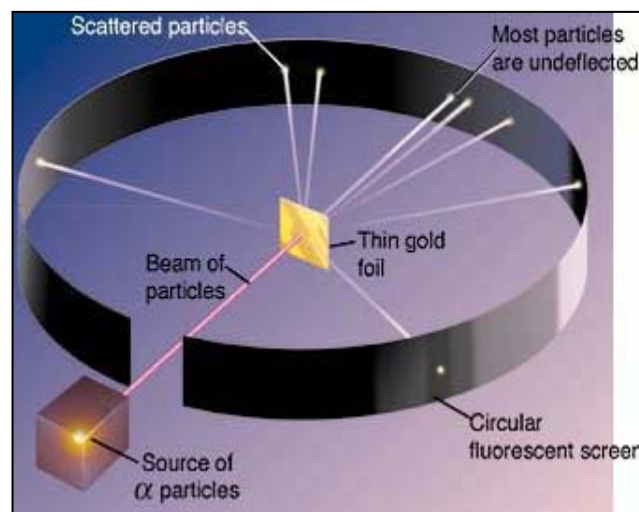


Using Spreadsheets to Demonstrate the Law of Large Numbers III

Introduction

A lot of high school students do not have a strong background in probability, statistics, and indirect measurement. Probably one of the most elegant demonstrations in the subject was done by Ernest Rutherford in 1911. In Rutherford's gold foil scattering experiment, he deduced the size and nature of the nucleus of the atom and postulated the nuclear model of the atom which is still used today in all science books. There are often



classroom activities that are designed to demonstrate Rutherford's experiment and indirect measurements, but they are often simplistic and crude. For example, dropping colored markers onto a target placed on sheet of paper or rolling marbles at gaps in a fence. In these activities they find the area of the object by

$$\frac{\text{Hits}}{\text{Trials}} = \frac{\text{Area of the Bullseye}}{\text{Area of the Target}} \quad (1)$$

This rearranges into:

$$\text{Area of the Bullseye} = \text{Area of the Target} \times \frac{\text{Hits}}{\text{Trials}} \quad (2)$$

Because students are unfamiliar with indirect measurements, they often erroneously think indirect measurements are not as accurate as direct measurements. And the technique of dropping markers and counting the number of hits is too tedious to disprove this fallacy to their satisfaction. They are surprised that you can sometimes measure more accurately by indirect measurement.

Computer simulation is an ideal solution. It is rapid and teaches the students to effectively use a spreadsheet and its mathematical tools. This provides more time for teacher-student interaction on the mathematics and science of the experiment. Using a spreadsheet emphatically demonstrates the principles and accuracy of indirect measurement. Finally, there are no safety issues involved.

Purpose

Students will use an Excel spreadsheet to

- Find the area of an object by indirect measurement, and
- Simulate Rutherford's Experiment.

In the process they will

- Avoid the tedium of counting hits and misses,
- Do many trials quickly,
- Use the spreadsheet to analyze the data,
- Examine and calculate the probabilities of hitting a specific small sample,
- Find the area of an object indirectly, and
- Recognize that using many trials gives better results (Law of Large Numbers).

Activity 1: Using the Spreadsheet to Determine the Area of an Object

In this activity students will accurately determine the area of an object by indirect measurement. Excel's random number generator will produce "hits" onto a target. The probability of a hit should be proportional to equation (2).

To run a simulation:

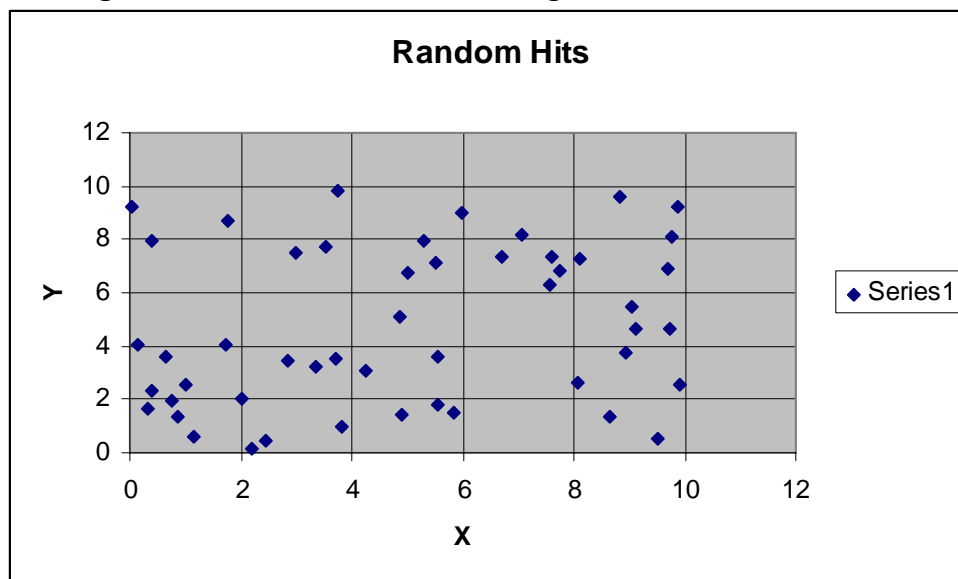
- Load Excel, go to "Tools" on the menu bar at the top, and select "Data Analysis." If you do not have "Data Analysis," you may have to download it by selecting "Add-Ins," then "Analysis ToolPak".
- Select "Random Number Generator" and hit "OK."
- On the "Random Number " table select:
 - ❖ Number of Variables: 2
 - ❖ Number of Random Numbers: 50
 - ❖ Distribution: Uniform
 - ❖ Parameters Between 0 and 10
 - ❖ Random Seed: (blank)
 - ❖ Check "Output Range:" and type $\$A\1 , or hit the red arrow and click on box A1 and hit the red arrow again.
 - ❖ Hit "OK."

Two columns, A and B, are produced, running from 1 to 50 numbers.

Constructing a scatter diagram:

- From the menu bar at the top, select the Graph Icon called "Chart Wizard."
- Select X-Y (Scatter) at the top of the page and hit "Next>."
- Excel will select the data. If not, drag over the data set or type "=Sheet 1!:\$A\$1:\$B\$50" into "Data Range."
- The chart will be produced when you hit "Next>."
- Do the following to the chart:
 - ❖ Chart title: Random Hits
 - ❖ Value (X) axis: X
 - ❖ Value (Y) axis: Y
 - ❖ Select the "Gridlines" tab at the top and click on major grid lines for the Value (X) axis and hit "Next>."
- Check "As Object in: Sheet 1" and "Finish."

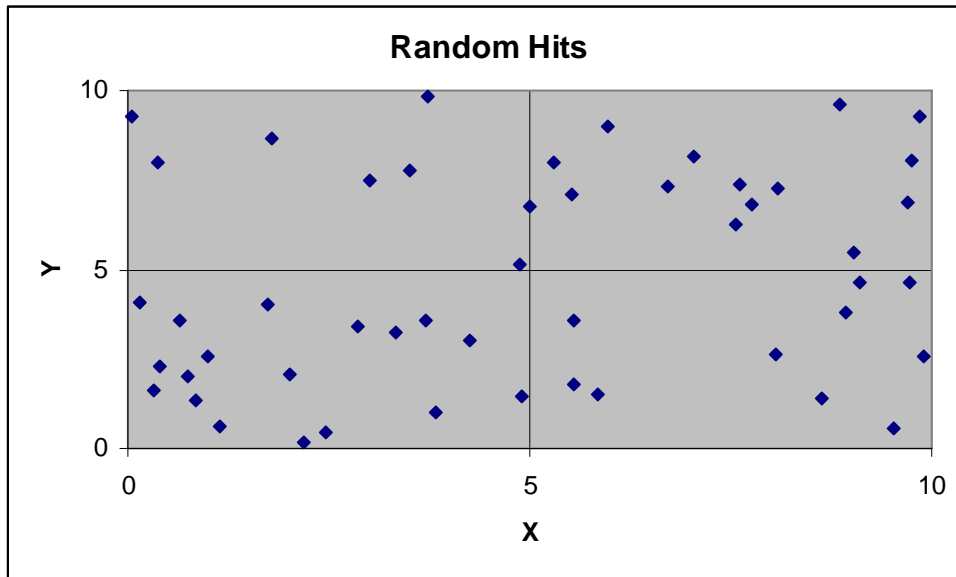
The scatter diagram will look like the following:



You will need to modify the diagram to make it more functional.

- Select "♦ Series 1" to the right by clicking on it and press the "Delete" key.
- Double click on the X-axis numbers and select the "Scale" tab.
- Enter the following:
 - ❖ Minimum: 0
 - ❖ Maximum: 10
 - ❖ Major unit: 5
 - ❖ Minor unit: 1
 - ❖ Hit "OK."
- Do the same for the Y-axis.

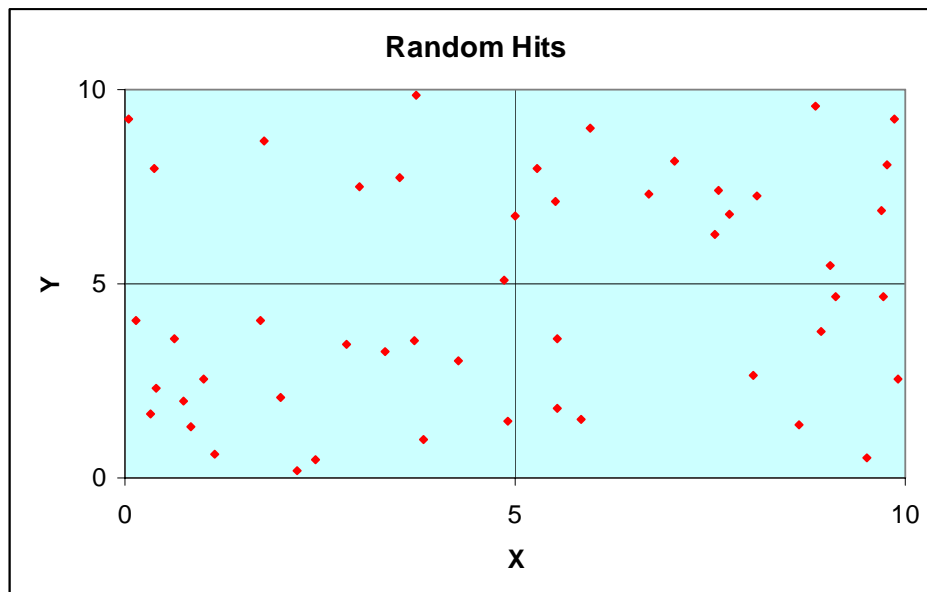
The scatter diagram will look like the following:



The next portion is strictly cosmetic and may be skipped, but students love it.

- Point the mouse over a blank area on the graph until it reads, "Plot Area." Then double left click. Select "Format Plot Area," and select a color for the background. Hit "OK" when you like the color. In the example a light blue is selected.
- Point the mouse over a data point until it reads, "Series 1..." Then double left click. Select the "Patterns," tab, and select an appropriate "Style." Change the "Size" to 2." And add color to the data points. For example, "Background: ■", and "Foreground: ■." Hit "OK" when you like the color.

The finished chart should look like the following:



Analyzing the Chart:

The quadrants will be designated as follows:

Q1	Q2
Q3	Q4

- Make a table in you spreadsheet starting in position M1, type "Q1", in N1, type "Q2," etc., ending at P1 with "Q4." Enter "Total" into column Q1.
- Count the number of hits in each quadrant, and enter the hits into its respective column.
- Under the total, select the "Σ" icon. When you hit "Enter<", the total should be 50. If not, recount your data.

Using the scatter diagram above, the first row of data will look like:

Q1	Q2	Q3	Q4	Total
8	14	18	10	50

Complete and hand in the following worksheet below before continuing.

Name_____ Period_____ Date_____

Spreadsheet on 50 Random Hits

Answer the following questions before continuing on.

1. What is a random number generator?
2. Do the 50 hits seem random? Explain why or why not.
3. What is the area in units of the entire graph?
4. What is the area in units of the each quadrant?
5. Use equation 1 and calculate the probability, p , of the hits in each quadrant.

Q1	Q2	Q3	Q4	Total

6. What is the sum of all the probabilities?

7. Based on the theoretical probability of $\frac{1}{4} = 0.25$, account for the discrepancy between this number and the values you got in #5.

8. The area of each box based on probability is $p \times 100$ (Equation 2). What is the area of Q1, Q2, Q3, and Q4?

9. How might you improve our "guess" of the area of the quadrant box?

Calculate the probabilities and enter them in a table in you spreadsheet starting in position R1, type "Q1", in S1, type "Q2," etc., ending at U1 with "Q4." Enter "Total" into column V1.

In row R2, type "=M2/\$Q2," and "Enter." Click on R2 and go to the button right corner until a "+" appears. Drag until V1 is filled. It should look like.

Q1	Q2	Q3	Q4	Total	Q1	Q2	Q3	Q4	Total
8	14	18	10	50	0.160	0.280	0.360	0.200	1.000

Hand this sheet in before going any further.

Applying the Law of Large Numbers:

Simulate 50 more hits

To run a simulation:

- Go to "Tools" on the menu bar at the top, and select "Data Analysis."
- Select "Random Number Generator" and hit "OK."
- On the "Random Number" table, hit "OK."

Two columns, A and B, are produced, running from 1 to 50 numbers. Count the data points in each quadrant and enter them into the Excel table like you did on the previous run.

Into M20, enter the "Σ" icon and drag to Q20. The two runs give a total of 100; therefore the total number in Q20 should be 100. Next, drag across R2 to V2, copy and paste into R20 to V20. This will give you the probability and area based on 100 hits.

Now rerun the random number generator, but change the Number of Random numbers to 100. This will give you 100 more hits.

Change the chart. Right click on a blank part of the chart, and select "Source Data." Go to the "Data Rang" tab and change the number at the far right from 50 to 100.

Count the number of hits per quadrant and enter them into the table.

As you do this, the data should look like the following:

Q1	Q2	Q3	Q4	Total	Q1	Q2	Q3	Q4	Total
49	49	52	50	200	0.245	0.245	0.260	0.250	1.000

Repeat the process until a total of 500 data points has been obtained.

Now do 500 more data points, for a total of 1,000 points.

Answer and then turn in the following worksheet before going any further.

Name_____ Period_____ Date_____

Spreadsheet on 1000 Random Hits

Answer the following questions before continuing on.

1. Do the 500 hits seem random? Explain why or why not.
2. Do the 1000 hits seem random? Explain why or why not.
3. What happened to the probabilities as more trials were added?
4. What is the area in units of the each quadrant after 500 trials? After 1000 trials?
5. Has your confidence increased as the number of trials increased?

6. What would happen in the number of trials was increased to 5000? To 10000?

7. Suppose the error of a direct measurement is estimated by using a ruler accurate to ± 0.1 mm. If the length of one side is 5.0 mm, the % error is:

$$(0.1/5.0) \times 100 = 2 \%$$

When we multiply the length by the width, the error increases due to two measurements.

Comment on the accuracy of an indirect measurement for the error of a box versus a direct measurement

Activity 2: Using a Spreadsheet to Find the Probability of Hitting an Object

In this activity, you will select a 1 x 1 box from the 10 x 10 graph and do the simulation to find the probability of hitting an object. By the way, what is the probability of a random hit?

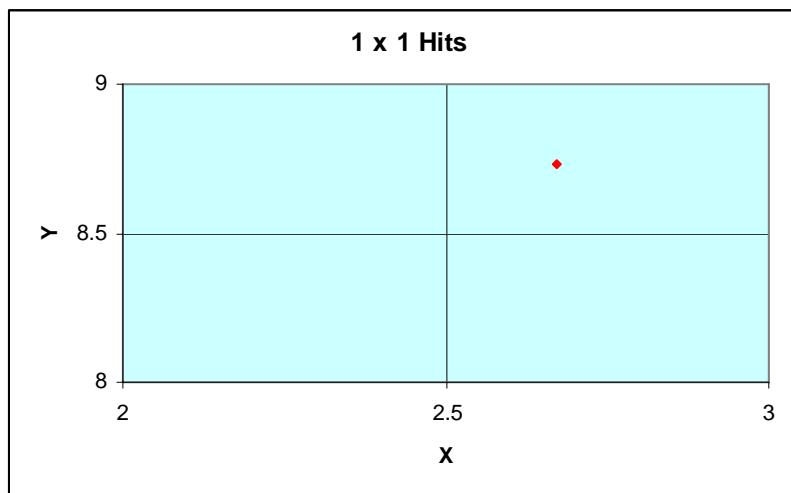
Select a 1 x 1 box. Some examples are shown below.

Example	X Range	Y Range
1	4 to 5	4 to 5
2	7 to 8	7 to 8
3	1 to 2	1 to 2
4	2 to 3	8 to 9

You will need to modify the diagram to make it more functional.

- Click on the Title at the top and change it to "1 x 1 Hits"
- Double click on the X-axis numbers and select the "Scale" tab.
- Enter the following:
 - ❖ Minimum: Your X range minimum
 - ❖ Maximum: Your X range maximum
 - ❖ Major unit: 0.5
 - ❖ Minor unit: 0.1
 - ❖ Adjust the "Crosses (Y) axis at" to the minimum Y value.
 - ❖ Hit "OK."
- Do the same for the Y-axis.

If you selected example 4, your graph should look like this.



You are now going to run 1000 hits.

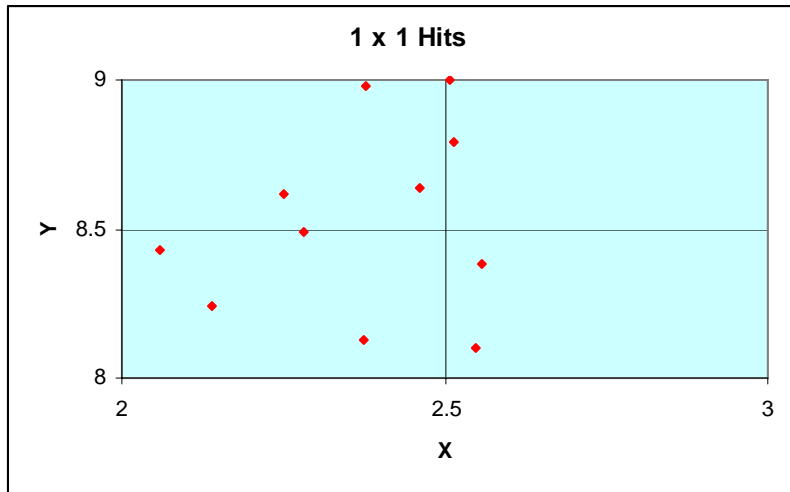
To run a simulation:

- Go to "Tools" on the menu bar at the top, and select "Data Analysis."
- Select "Random Number Generator" and hit "OK."
- On the "Random Number" table, hit "OK."

Now rerun the random number generator, but change the Number of Random numbers to 1000. This will give you 1000 more hits.

Change the chart. Right click on a blank part of the chart, and select "Source Data." Go to the "Data Range" tab and change the number at the far right from 100 to 1000.

Your first simulation should look like this:



Place the data in a table similar to the one you did for the first activity. Change the total to 1000. Have the second total be the sum of the 4 probabilities to the left of "Total." The table should look like the following:

Q1	Q2	Q3	Q4	Total	Q1	Q2	Q3	Q4	Total
3	2	4	2	1000	0.003	0.002	0.004	0.002	0.011

After 10000 simulations, your table should look like the table on the next page:

Principles of Probability: Using Spreadsheets to Demonstrate the Law of Large Numbers III

Q1	Q2	Q3	Q4	Total	Q1	Q2	Q3	Q4	Total
3	2	4	2	1000	0.0030	0.0020	0.0040	0.0020	0.0110
5	1	4	3	1000	0.0050	0.0010	0.0040	0.0030	0.0130
1	3	1	5	1000	0.0010	0.0030	0.0010	0.0050	0.0100
2	1	6	2	1000	0.0020	0.0010	0.0060	0.0020	0.0110
2	4	2	3	1000	0.0020	0.0040	0.0020	0.0030	0.0110
2	0	1	2	1000	0.0020	0.0000	0.0010	0.0020	0.0050
5	1	0	3	1000	0.0050	0.0010	0.0000	0.0030	0.0090
1	3	4	3	1000	0.0010	0.0030	0.0040	0.0030	0.0110
1	3	2	1	1000	0.0010	0.0030	0.0020	0.0010	0.0070
3	5	3	2	1000	0.0030	0.0050	0.0030	0.0020	0.0130
25	23	27	26	10000	0.0025	0.0023	0.0027	0.0026	0.0101

Answer the following worksheet and turn it in when finished.

Name_____ Period_____ Date_____

Activity 2 Spreadsheet on 1 x 1 Squares.

Answer the following questions before continuing on.

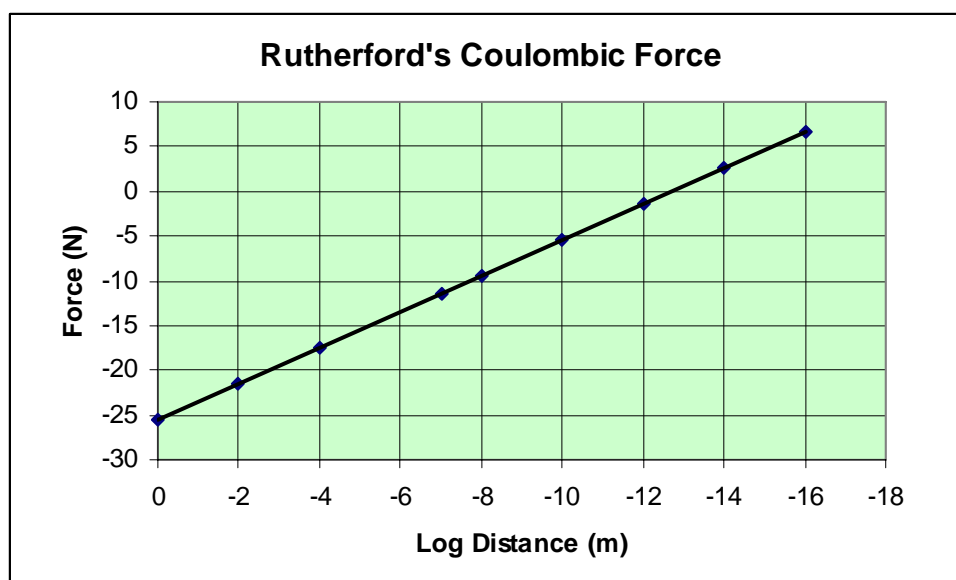
1. Find the probability of a 1 x 1 hit at 1000, 5000, and 10000 trials. Do the hits seem random? Explain why or why not.
2. Find the probability of a 0.25 x 0.25 hit at 1000, 5000, and 10000 trials. Do the hits seem random? Explain why or why not.
3. Does the size of the area seem to make any difference?
4. What happened to the probabilities as more trials were added?
5. Calculate the area of the 1 x 1 and 0.25 x 0.25 squares based on probability.

Activity 3: Using a Spreadsheet to Simulate Rutherford's Scattering Experiment

Ernest Rutherford, Hans Geiger, and Ernest Marsden examined over 100,000 alpha particle scatterings, and that's a lot of data. From the scattering angles, trajectory paths, masses of Au and alpha particles, and the velocity of the alpha particles, Rutherford concluded that the scattering was due to electrostatic forces alone. From Coulomb's Law,

$$F_C = \frac{kZ_1Z_2e^2}{r^2} \quad (3)$$

the force versus distance may be calculated. This assumes that the particles are point charges and Z_1 and Z_2 are the charges of the alpha particles (+2) and gold (+79) respectively. Advanced students may use Excel to construct a Coulomb's Law graph of distance versus force. Since the force varies dramatically with distance a log plot gives a better picture.



Rutherford did his scattering calculations by assuming that there was no electrostatic repulsion at first and that the force was only applicable when the alpha particle approached the nucleus. We see from the graph that this distance is close to around 10^{-13} to 10^{-14} meters. Since the size of an atom is around 10^{-10} meters, Rutherford concluded that the Thomson plumb pudding model was incorrect. He found that around 1 alpha particle in 8000 was deflected at a large angle. Scattering means a hit! Since we have a setup for measuring hits, let's see if we can simulate his historic experiment.

Preliminary Questions

1. What is the probability of hitting a gold atom?
2. What size square of a 10 x 10 grid would this be? Hint: $10 \times 10 = 100$ square units. The probability is around 0.0001. Use equation 2 to calculate this.
3. What is the distance on one side of the square?

Procedure

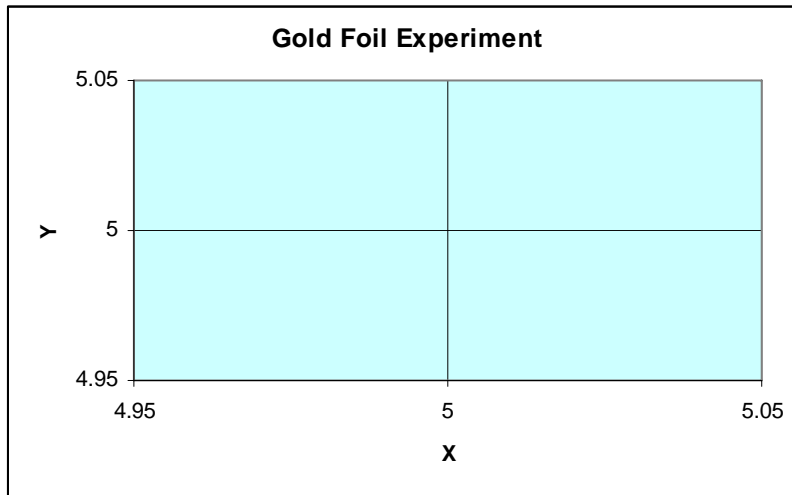
Assume that the nucleus is in the center of the square 10 x 10 grid. Select a 0.1 x 0.1 box near or at the center. Write down the X and Y values that describe this box.

X Range _____ Y Range _____

You will need to modify the diagram to make it more functional.

- Click on the Title at the top and change it to "Gold Foil Experiment"
- Double click on the X-axis numbers and select the "Scale" tab.
- Enter the following:
 - ❖ Minimum: Your X range minimum
 - ❖ Maximum: Your X range maximum
 - ❖ Major unit: 0.05
 - ❖ Minor unit: 0.01
 - ❖ Adjust the "Crosses (Y) axis at" to the minimum Y value
 - ❖ Hit "OK."
- Do the same for the Y-axis.

The nucleus of the gold atom looks like the following:



You are now going to run 10,000 hits.

To run a simulation:

- Go to "Tools" on the menu bar at the top, and select "Data Analysis."
- Select "Random Number Generator" and hit "OK."
- On the "Random Number" table, hit "OK."

Now rerun the random number generator, but change the Number of Random numbers to 10000. This will give you 10000 more hits.

Change the chart. Right click on a blank part of the chart, and select "Source Data." Go to the "Data Range" tab and change the number at the far right from 1000 to 10000.

Run 100,000 simulations and record your results.

Your final results should look like the following:

Q1	Q2	Q3	Q4	Total	Q1	Q2	Q3	Q4
4	5	4	5	150000	2.67E-05	3.33E-05	2.67E-05	3.33E-05

The total probability from the above data is $p = (18/150000) = 1.2 \times 10^{-5}$. Based on the Rutherford experimental scattering, $p = 1/8000 = 1.25 \times 10^{-5}$. Calculate your probability, showing your data and include it in the report below.

10. Suppose we use circles rather than squares. What effect would this have on the simulation?

11. How might a much smaller number of experimental points have affected Rutherford's conclusions?

12. Do you suppose that there is a limited number of trials that will give you good data and an increasing number that will only marginally improve the results? Look up the Central Mean Theorem to help with your answer.

13. Who is this guy and in what country was he born?

